THERE IS NO HERON TRIANGLE WITH THREE RATIONAL MEDIANS

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ABSTRACT. The subject of this article is the proof that the Heron triangle with three integer medians does not exist. The article provides proofs of three lemmas. As a result, the infinite descent method proved that the Heron triangle with three integer medians does not exist. In the process of proving Lemma 3, the fact was used that there is no right-angled Heron triangle with three integer medians. And then it is proved that there is no Heron triangle with three natural medians.
The relevance of this article lies in the fact that the problem under study is one of the unsolved problems of number theory.

KEY WORDS. Heron triangles; Integer triangle; Number theory; geometry.

CLASSIFICATION NUMBERS: MSC: 11R04, 14G99, 11D99

1. INTRODUCTION

The problem: *Does a triangle with integer sides, integer medians and integer area exist?* [1, 2, 3, 4, 5, 6, 7].
It is known that there are triangles with integer sides and medians. For example, the smallest of these triangles has sides and medians (136, 170, 174) and (158, 131, 127), respectively.
In this article, we prove the theorem that there is no triangle with three integer sides, three integer medians and an integer area. To do this, the following three lemmas with proofs and an auxiliary axiom are given at the beginning (with very important explanations, added in the latest version of the proof of Lemma 3):

Lemma 1 *For any triangle with rational sides and medians, there is another, but not similar triangle with rational sides and medians.*

Lemma 2. *If at least one median of a triangle with integer sides and medians is a multiple of 3, then all its medians are multiples of 3.*

Lemma 3. *If we assume that there is a triangle with integer sides, medians and area, then at least one of its medians must be multiple of 3.*

Theorem. There is no Heron triangle with three integer medians.
Using the results of the above three lemmas, we prove the theorem by the method of infinite descent.

2. PROOFS

Proof of Lemma 1.
Here it is proved that triangles with integer sides and medians exist only and only in pairs (like "twins") – one (any) of which follows from the other, and these two triangles are not similar triangles between themselves.

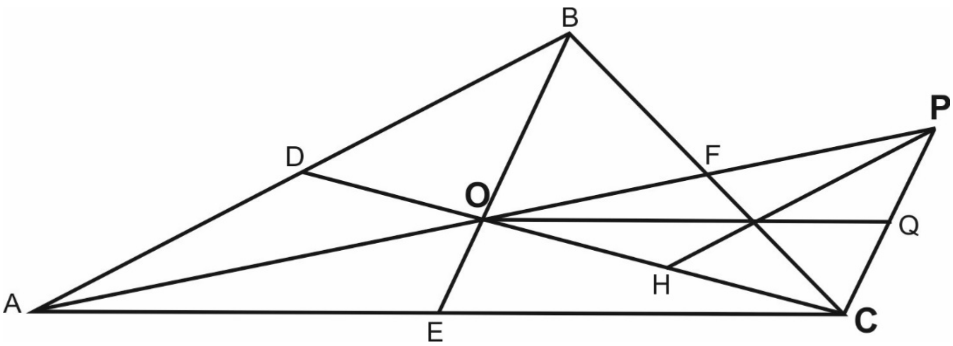

Fig1

Here (for Fig1): $AB = c,\ BC = a,\ \ AC = b,\ AF = m_a,\ BE = m_b,\ DC = m_c,$
$OP = \frac{2}{3}m_a,\ \ CP = \frac{2}{3}m_b, OC = \frac{2}{3}m_c\ \ OQ = \frac{1}{2}b,\ HP = \frac{1}{2}c, CF = \frac{1}{2}a$
The last six equalities are obtained by the results of Lemma 1.
Assume that the sides and medians of the triangle ΔABC are rational (Fig1). Using the triangle ΔABC, we construct the triangle ΔOPC.
To do this, draw (starting from point C) $CP \| OB$ to the intersection with the continuation of the median $AF = m_a$ at point P.
It turns out that the triangles ΔAPC and ΔAOE are similar and
$CP: EO = AP: AO = AC: AE = 2: 1$
Taking into account the properties of the medians ΔABC for the sides of the triangle ΔOPC, we obtain

$$CP = OB = \frac{2}{3}m_b, \quad OC = \frac{2}{3}m_c, \quad OP = OA = \frac{2}{3}m_a \tag{1}$$

For the medians of the triangle ΔOPC, it turns out

$$FC = \frac{1}{2}BC = \frac{1}{2}a, \quad OQ = \frac{1}{2}AC = \frac{1}{2}b, \quad HP = \frac{1}{2}AB = \frac{1}{2}c \tag{2}$$

In other words, a triangle ΔOPC is constructed by parallel displacements of $\frac{2}{3}$ of the segments of the medians of triangle ΔABC. As for medians of triangle ΔOPC they are constructed by parallel displacements of $\frac{1}{2}$ parts of ΔABC triangle's sides. This means that all sides and medians of the ΔOPC triangle are also rational.

The triangles ΔABC and ΔOPC are not similar to each other. The sides of these triangles are rational and do not have the similarity property. The ratio of the areas of similar triangles should be equal to the square of the similarity coefficient.

In our case (Fig1) the ratio of the areas of the triangles is

$$\frac{A_{\Delta OPC}}{A_{\Delta ABC}} = \frac{1}{3} \tag{3}$$

which is not the square of rational number.

Lemma 1 is proved.

Note 1. Taking into account (3), we obtain equality (12).

Proof of Lemma 2.

Let's write down the formulas of dependence between the sides and medians of triangles:

$$\begin{cases} a = \frac{2}{3}\sqrt{2m_b^2 + 2m_c^2 - m_a^2} \\ b = \frac{2}{3}\sqrt{2m_a^2 + 2m_c^2 - m_b^2} \\ c = \frac{2}{3}\sqrt{2m_a^2 + 2m_b^2 - m_c^2} \end{cases} \Rightarrow \begin{cases} m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \\ m_b = \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2} \\ m_c = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2} \end{cases} \Rightarrow \begin{cases} m_a = \frac{1}{2}\sqrt{3b^2 + 3c^2 - (a^2 + b^2 + c^2)} \\ m_b = \frac{1}{2}\sqrt{3a^2 + 3c^2 - (a^2 + b^2 + c^2)} \\ m_c = \frac{1}{2}\sqrt{3a^2 + 3b^2 - (a^2 + b^2 + c^2)} \end{cases} \tag{4}$$

It is obvious from these three formulas that if at least one of the medians is a multiple of 3, then

$$a^2 + b^2 + c^2 \equiv 0(mod3) \tag{5}$$

This means that all three medians are multiples of 3.

Lemma 2 is proved.

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I PART OF THE CALCULATIONS – CALCULATIONS WITH MEDIANS

CHARACTERISTICS OF THREE (Fig2, Fig3, Fig4) TRIANGLES

In addition to Fig1, we examine three more figures.

In the proof of Lemma 1, we have constructed the ΔOPC triangle (Fig1).

Using the ΔOPC triangle, three more triangles are constructed (the vertices of the ΔOPC triangle in all three figures are preserved and indicated in large letters).

Parameters of Fig2. Taking the vertex P as the intersection point of the medians, the triangle ΔOMC is constructed (Fig2).

It will be useful if we note that the ΔOMC triangle has one median (HM) equal to $\frac{3}{2}c$, two medians (OT, CV) equal to two medians of the ΔABC triangle (m_a, m_b). The two sides OM and CM of the triangle ΔOMC are not investigated in this article.

If we construct a triangle from the medians $\left(m_a, m_b, \frac{3}{2}c\right)$ of the ΔOMC triangle, then the formula for the area of the resulting triangle (let's denote ΔT_2) will be as follows

$$A_{\Delta T_2} = \frac{1}{4}\sqrt{\left(m_a + m_b + \frac{3}{2}c\right)\left(m_a + m_b - \frac{3}{2}c\right)\left(m_a - m_b + \frac{3}{2}c\right)\left(-m_a + m_b + \frac{3}{2}c\right)} \qquad (6)$$

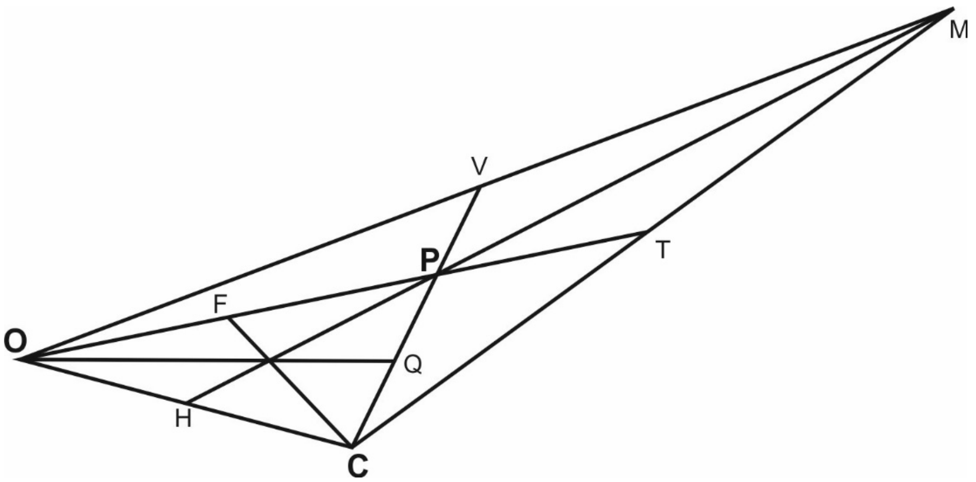

Fig2

Here (for Fig2): $CP = \frac{2}{3}m_b,\ CV = m_b,\ OP = \frac{2}{3}m_a,\ OT = m_a,\ OC = \frac{2}{3}m_c,$ $HM = \frac{3}{2}c, A_{\Delta OMC} = 3A_{\Delta OPC}$.

Parameters of Fig3. Taking the vertex O as the intersection point of the medians, the triangle ΔNPC is constructed (Fig3).

It will be useful if we note that the ΔNPC triangle has one median (NQ) equal to $\frac{3}{2}b$, two medians (PW, SU) equal to two medians of the ΔABC triangle (m_a, m_c). The two sides NP and NC of the triangle ΔNPC are not investigated in this article.

If we construct a triangle from the medians $\left(m_a, m_c, \frac{3}{2}b\right)$ of the ΔNPC triangle, then the formula for the area of the resulting triangle (let's denote ΔT_3) will be as follows

$$A_{\Delta T_3} = \frac{1}{4}\sqrt{\left(m_a + m_c + \frac{3}{2}b\right)\left(m_a + m_c - \frac{3}{2}b\right)\left(m_a - m_c + \frac{3}{2}b\right)\left(-m_a + m_c + \frac{3}{2}b\right)} \quad (7)$$

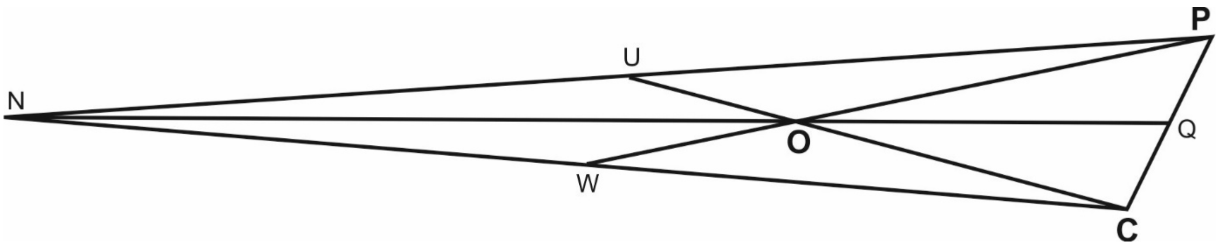

Fig3

Here (for Fig3): $CO = \frac{2}{3}m_c$, $CU = m_c$, $PO = \frac{2}{3}m_a$, $PW = m_a$, $PC = \frac{2}{3}m_b$, $NQ = \frac{3}{2}b$, $A_{\Delta NPC} = 3A_{\Delta OPC}$.

Parameters of Fig4. Taking the vertex C as the intersection point of the medians, the triangle ΔOPL is constructed (Fig4).

It will be useful if we note that the ΔOPL triangle has one median (FL) equal to $\frac{3}{2}a$, two medians (PR, OK) equal to two medians of the ΔABC triangle (m_b, m_c).

The two sides OL and PL of the triangle ΔOPL are not investigated in this article.

If we construct a triangle from the medians $\left(m_b, m_c, \frac{3}{2}a\right)$ of the ΔNPC triangle, then the formula for the area of the resulting triangle (let's denote ΔT_4) will be as follows

$$A_{\Delta T_4} = \frac{1}{4}\sqrt{\left(m_b + m_c + \frac{3}{2}a\right)\left(m_b + m_c - \frac{3}{2}a\right)\left(m_b - m_c + \frac{3}{2}a\right)\left(-m_b + m_c + \frac{3}{2}a\right)} \quad (8)$$

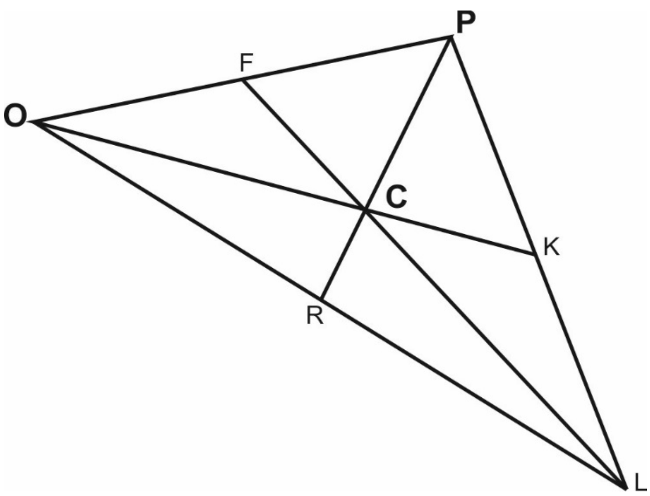

Fig4

Here (for Fig4): $PC=\frac{2}{3}m_b,\ PR=m_b,\ OC=\frac{2}{3}m_c,\ OK=m_c,\ OP=\frac{2}{3}m_a,$ $FL=\frac{3}{2}a,\ A_{\Delta OPL}\ =3A_{\Delta OPC}$.

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If we construct a triangle from the medians (m_a, m_b, m_c) of the ΔABC triangle, then the formula for the area of the resulting triangle (let's denote ΔT_1) will be as follows

$$A_{\Delta T_1}\ =\frac{1}{4}\sqrt{(m_a+m_b+m_c)(m_a+m_b-m_c)(m_a-m_b+m_c)(-m_a+m_b+m_c)} \qquad (9)$$

Let's write down the formula for the area of the triangle ΔABC.

$$A_{\Delta ABC}\ =\frac{1}{3}\sqrt{(m_a+m_b+m_c)(m_a+m_b-m_c)(m_a-m_b+m_c)(-m_a+m_b+m_c)} \qquad (10)$$

Taking into account (9) and (10), we get

$$A_{\Delta T_1}\ =\frac{3}{4}A_{\Delta ABC} \qquad (11)$$

After studying Lemma 1, and taking into account (3) we learn that

$$A_{\Delta OMC}\ =A_{\Delta NPC}\ =A_{\Delta OPL}\ =A_{\Delta ABC}\ =3A_{\Delta OPC} \qquad (12)$$

Taking into account (12), we get

$$A_{\Delta T_2}\ =A_{\Delta T_3}\ =A_{\Delta T_4}\ =A_{\Delta T_1} \qquad (13)$$

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Proof of Lemma 3.

Suppose there is a triangle ΔABC with integer sides, medians and area (Fig1), where

$$(a,b,c,m_a,m_b,m_c,A_{\Delta ABC})=1. \qquad (14)$$

Note that when executing (14), the following is performed:

$$(a,b,c,m_a,m_b,m_c,S)=1 \qquad (14^*)$$

Note that taking into account (4), $(a,b,c)\neq 4$ is fulfilled. Let $c\equiv 0(mod2^k)$, and for the maximum value of k, $k_{max}=1$ is performed.

Taking into account (4), we know that if $(a,b,c)=M\in N$ is true, where $M\neq\{2,3\}$, then it is also true for medians $(m_a,m_b,m_c)=M$, and as a result we get $(a,b,c,m_a,m_b,m_c)=M$, hence $(a,b,c,m_a,m_b,m_c,S_{\Delta ABC})=M$. (14**)

Let's prove that the medians of the triangle ΔABC are multiples of 3.

From two equations (9) and (6) we will make a system of equations.

$$\begin{cases} A_{\Delta T_1} = \frac{1}{4}\sqrt{(m_a + m_b + m_c)(m_a + m_b - m_c)(m_a - m_b + m_c)(-m_a + m_b + m_c)} \\ A_{\Delta T_2} = \frac{1}{4}\sqrt{\left(m_a + m_b + \frac{3}{2}c\right)\left(m_a + m_b - \frac{3}{2}c\right)\left(m_a - m_b + \frac{3}{2}c\right)\left(-m_a + m_b + \frac{3}{2}c\right)} \end{cases} \quad (15)$$

NOTE. In (15) we arbitrarily took ΔT_1 and ΔT_2. We could take ΔT_1 and ΔT_3, or ΔT_1 and ΔT_4.

Taking into account (13), we denote

$$A_{\Delta T_1} = A_{\Delta T_2} = S$$

$$\begin{cases} S = \frac{1}{4}\sqrt{(m_a + m_b + m_c)(m_a + m_b - m_c)(m_a - m_b + m_c)(-m_a + m_b + m_c)} \\ S = \frac{1}{4}\sqrt{\left(m_a + m_b + \frac{3}{2}c\right)\left(m_a + m_b - \frac{3}{2}c\right)\left(m_a - m_b + \frac{3}{2}c\right)\left(-m_a + m_b + \frac{3}{2}c\right)} \end{cases} \quad (15^*)$$

We multiply both sides by 4 and get

$$\begin{cases} 4S = \sqrt{(m_a + m_b + m_c)(m_a + m_b - m_c)(m_a - m_b + m_c)(-m_a + m_b + m_c)} \\ 4S = \sqrt{\left(m_a + m_b + \frac{3}{2}c\right)\left(m_a + m_b - \frac{3}{2}c\right)\left(m_a - m_b + \frac{3}{2}c\right)\left(-m_a + m_b + \frac{3}{2}c\right)} \end{cases}$$

We square both sides and get

$$\begin{cases} (4S)^2 = (m_a + m_b + m_c)(m_a + m_b - m_c)(m_a - m_b + m_c)(-m_a + m_b + m_c) \\ (4S)^2 = \left(m_a + m_b + \frac{3}{2}c\right)\left(m_a + m_b - \frac{3}{2}c\right)\left(m_a - m_b + \frac{3}{2}c\right)\left(-m_a + m_b + \frac{3}{2}c\right) \end{cases}$$

$m_a + m_b$ are written in parentheses $(m_a + m_b)$

$$\begin{cases} (4S)^2 = \big((m_a + m_b) + m_c\big)\big((m_a + m_b) - m_c\big)(m_c + m_a - m_b)(m_c - m_a + m_b) \\ (4S)^2 = \left((m_a + m_b) + \frac{3}{2}c\right)\left((m_a + m_b) - \frac{3}{2}c\right)\left(\frac{3}{2}c + m_a - m_b\right)\left(\frac{3}{2}c - m_a + m_b\right) \end{cases}$$

$m_a - m_b$ are written in parentheses $(m_a - m_b)$

$$\begin{cases} (4S)^2 = \big((m_a + m_b) + m_c\big)\big((m_a + m_b) - m_c\big)\big(m_c + (m_a - m_b)\big)\big(m_c - (m_a - m_b)\big) \\ (4S)^2 = \left((m_a + m_b) + \frac{3}{2}c\right)\left((m_a + m_b) - \frac{3}{2}c\right)\left(\frac{3}{2}c + (m_a - m_b)\right)\left(\frac{3}{2}c - (m_a - m_b)\right) \end{cases}$$

Let's take into account the well-known algebraic identities:

$$\begin{cases} \big((m_a + m_b) + m_c\big)\big((m_a + m_b) - m_c\big) \equiv (m_a + m_b)^2 - m_c^2 \\ \big(m_c + (m_a - m_b)\big)\big(m_c - (m_a - m_b)\big) \equiv m_c^2 - (m_a - m_b)^2 \\ \left((m_a + m_b) + \frac{3}{2}c\right)\left((m_a + m_b) - \frac{3}{2}c\right) \equiv (m_a + m_b)^2 - \left(\frac{3}{2}c\right)^2 \\ \left(\frac{3}{2}c + (m_a - m_b)\right)\left(\frac{3}{2}c - (m_a - m_b)\right) \equiv \left(\left(\frac{3}{2}c\right)^2 - (m_a - m_b)^2\right) \end{cases}$$

We get

$$\begin{cases} (4S)^2 = \big((m_a + m_b)^2 - m_c^2\big)\big(m_c^2 - (m_a - m_b)^2\big) \\ (4S)^2 = \left((m_a + m_b)^2 - \left(\frac{3}{2}c\right)^2\right)\left(\left(\frac{3}{2}c\right)^2 - (m_a - m_b)^2\right) \end{cases} \quad (16)$$

Let's denote some expressions as follows (for convenience of calculations):

$$\begin{cases}(m_a + m_b)^2 - m_c^2 = x \\ m_c^2 - (m_a - m_b)^2 = y \\ \left(\frac{3}{2}c\right)^2 - m_c^2 = \delta \\ (m_a + m_b)^2 - \left(\frac{3}{2}c\right)^2 = x - \delta \\ \left(\frac{3}{2}c\right)^2 - (m_a - m_b)^2 = y + \delta\end{cases} \quad (17)$$

Replace in (16)

$$\begin{cases}(4S)^2 = xy \\ (4S)^2 = (x - \delta)(y + \delta)\end{cases} \quad (18)$$

We get

$$xy = (x - \delta)(y + \delta) \Rightarrow xy = xy + x\delta - \delta y - \delta^2 \Rightarrow 0 = (x - y - \delta)\delta \quad (19)$$

Let's solve equation (19)

Either $\delta = 0,$ (20)

either $x - y - \delta = 0.$ (21)

If $\delta = 0$, then $\delta = \left(\frac{3}{2}c\right)^2 - m_c^2 = 0.$

$$\left(\frac{3}{2}c\right)^2 - m_c^2 = 0 \Rightarrow \frac{3}{2}c - m_c = 0 \Rightarrow c = \frac{2}{3}m_c \Rightarrow \frac{2}{3}m_c = \frac{2}{3}\sqrt{2m_a^2 + 2m_b^2 - m_c^2} \Rightarrow$$

$$\Rightarrow m_c = \sqrt{2m_a^2 + 2m_b^2 - m_c^2} \Rightarrow m_c^2 = 2m_a^2 + 2m_b^2 - m_c^2 \Rightarrow$$

$$\Rightarrow m_c^2 = m_a^2 + m_b^2 \quad (22)$$

And this (22) is impossible. Because we took ΔT_1 and ΔT_2 in (15) arbitrarily. We could take ΔT_1 and ΔT_3, or ΔT_1 and ΔT_4.

If we took ΔT_1 and ΔT_3, we would get $m_b^2 = m_a^2 + m_c^2$.

If we took ΔT_1 and ΔT_4, we would get $m_a^2 = m_b^2 + m_c^2$.

If

$$\begin{cases}m_c^2 = m_a^2 + m_b^2 \\ m_b^2 = m_a^2 + m_c^2, \\ m_a^2 = m_b^2 + m_c^2\end{cases}$$

then $m_a = m_b = m_c = 0$.

According to the conditions of the problem $m_a, m_b, m_c \neq 0$

If in (19) $x-y-\delta=0$, then $y=x-\delta$. (23)

Taking into account (23) in the first equation (18) we will replace $y=x-\delta$, and in the second equation we will replace $x-\delta=y$.

$$\begin{cases}(4S)^2=xy\\(4S)^2=(x-\delta)(y+\delta)\end{cases}\Rightarrow\begin{cases}(4S)^2=x(x-\delta)\\(4S)^2=y(y+\delta)\end{cases} \quad (24)$$

As a result (24) and (17) instead of the system of equations (16), we get the following system of equations (25).

$$\begin{cases}(4S)^2=((m_a+m_b)^2-m_c^2)\left((m_a+m_b)^2-\left(\frac{3}{2}c\right)^2\right)\\(4S)^2=(m_c^2-(m_a-m_b)^2)\left(\left(\frac{3}{2}c\right)^2-(m_a-m_b)^2\right)\end{cases} \quad (25)$$

Let's write it like this

$$\begin{cases}((m_a+m_b)^2-m_c^2)\left((m_a+m_b)^2-\left(\frac{3}{2}c\right)^2\right)=(4S)^2\\(m_c^2-(m_a-m_b)^2)\left(\left(\frac{3}{2}c\right)^2-(m_a-m_b)^2\right)=(4S)^2\end{cases}$$

$(4S)^2$ move to the left side

$$\begin{cases}((m_a+m_b)^2-m_c^2)\left((m_a+m_b)^2-\left(\frac{3}{2}c\right)^2\right)-(4S)^2=0\\(m_c^2-(m_a-m_b)^2)\left(\left(\frac{3}{2}c\right)^2-(m_a-m_b)^2\right)-(4S)^2=0\end{cases}$$

Let's open some brackets, and we get

$$\begin{cases}(m_a+m_b)^4-(m_a+m_b)^2\left(\frac{3}{2}c\right)^2-m_c^2(m_a+m_b)^2+m_c^2\left(\frac{3}{2}c\right)^2-(4S)^2=0\\m_c^2\left(\frac{3}{2}c\right)^2-m_c^2(m_a-m_b)^2-(m_a-m_b)^2\left(\frac{3}{2}c\right)^2+(m_a-m_b)^4-(4S)^2=0\end{cases}$$

$$\begin{cases}(m_a+m_b)^4-\left(m_c^2+\left(\frac{3}{2}c\right)^2\right)(m_a+m_b)^2+\left(\frac{3}{2}cm_c\right)^2-(4S)^2=0\\(m_a-m_b)^4-\left(m_c^2+\left(\frac{3}{2}c\right)^2\right)(m_a-m_b)^2+\left(\frac{3}{2}cm_c\right)^2-(4S)^2=0\end{cases} \quad (26)$$

Let's denote it like this

$$\begin{cases}(m_a+m_b)^2=z\\(m_a-m_b)^2=w\end{cases} \quad (27)$$

We get two equivalent equations:

$$z^2-\left(m_c^2+\left(\frac{3}{2}c\right)^2\right)z+\left(\frac{3}{2}cm_c\right)^2-(4S)^2=0 \quad (28)$$

$$w^2-\left(m_c^2+\left(\frac{3}{2}c\right)^2\right)w+\left(\frac{3}{2}cm_c\right)^2-(4S)^2=0 \quad (29)$$

The first option:

Is such equality possible?

$$z = w \Rightarrow (m_a + m_b)^2 = (m_a - m_b)^2 \Rightarrow$$

$$\Rightarrow (m_a + m_b)^2 - (m_a - m_b)^2 = 0 \Rightarrow 4m_a m_b = 0 \quad (30)$$

According to the conditions of the problem $m_a m_b \neq 0$.

The second option:

In this case, it is enough to solve one of the two equations (let's explore z). Obviously, we have a quadratic equation with coefficients

$$m_c^2 + \left(\frac{3}{2}c\right)^2 \text{ и } \left(\frac{3}{2}cm_c\right)^2 - (4S)^2, \quad (31)$$

and by requirement (26) has two roots $(m_a + m_b)^2$ and $(m_a - m_b)^2$.

That is, if

$$z^2 - \left(m_c^2 + \left(\frac{3}{2}c\right)^2\right)z + \left(\frac{3}{2}cm_c\right)^2 - (4S)^2 = 0, \quad (32)$$

then $z_1 = (m_a + m_b)^2$ и $z_2 = (m_a - m_b)^2$. (33)

By Vieta 's theorem (the sum of the roots)

$$(m_a + m_b)^2 + (m_a - m_b)^2 = m_c^2 + \left(\frac{3}{2}c\right)^2 \Rightarrow$$

$$\Rightarrow 2m_a^2 + 2m_b^2 - m_c^2 = \frac{9}{4}c^2 \Rightarrow$$

$$\Rightarrow \boldsymbol{c \equiv \frac{2}{3}\sqrt{2m_a^2 + 2m_b^2 - m_c^2}} \quad \underline{\textbf{(identity)}} \quad (34)$$

The last formula (34) – the formula (identity) of the sides of the triangle through the medians.

By Vieta 's theorem (the product of roots)

$$(m_a + m_b)^2(m_a - m_b)^2 = \left(\frac{3}{2}cm_c\right)^2 - (4S)^2 \Rightarrow$$

$$\Rightarrow (m_a^2 - m_b^2)^2 + (4S)^2 = \left(\frac{3}{2}cm_c\right)^2 \quad (35)$$

Due to the arbitrariness of the choice from options (13), we can perform calculations for all $A_{\Delta T_1} = A_{\Delta T_3}$, $A_{\Delta T_1} = A_{\Delta T_4}$, as it was in (15), and we get:

$$\begin{cases} (m_a^2 - m_b^2)^2 + (4S)^2 = \left(\frac{3}{2}cm_c\right)^2 \\ (m_a^2 - m_c^2)^2 + (4S)^2 = \left(\frac{3}{2}bm_b\right)^2 \\ (m_b^2 - m_c^2)^2 + (4S)^2 = \left(\frac{3}{2}am_a\right)^2 \end{cases} \quad (36)$$

Remark. These equalities (36) are **identities**, for any triangle.

II PART OF CALCULATIONS – CALCULATIONS WITH THE SIDES.

If we perform the same calculations (which we performed in part I of the calculations) with the sides of the triangle ΔABC, then we get the following three equalities:

$$\begin{cases}\left(\frac{3}{4}(a^2-b^2)\right)^2+(4S)^2=\left(\frac{3}{2}cm_c\right)^2\\ \left(\frac{3}{4}(a^2-c^2)\right)^2+(4S)^2=\left(\frac{3}{2}bm_b\right)^2\\ \left(\frac{3}{4}(b^2-c^2)\right)^2+(4S)^2=\left(\frac{3}{2}am_a\right)^2\end{cases} \quad (37)$$

It is (37) received like this:

Instead of the ΔABC triangle, we take the ΔOPC triangle. As a result, the medians m_a, m_b, m_c were replaced by $\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c$. The sides a, b, c were replaced by $\frac{2}{3}m_a, \frac{2}{3}m_b, \frac{2}{3}m_c$. Taking into account condition (3), we write $\frac{S_1}{S}=\frac{1}{3}$. Here S is the area of the triangle, the sides of which are the medians of the triangle ΔABC (formula (15*)). S_1 is the area of a triangle whose sides are the medians of the ΔOPC triangle.

$$\begin{cases}S_1=\frac{1}{4}\sqrt{\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}c\right)\left(\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}c\right)\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}c\right)\left(-\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}c\right)}\\ S_1=\frac{1}{4}\sqrt{\left(\frac{1}{2}a+\frac{1}{2}b+\frac{3}{2}\cdot\frac{2}{3}m_c\right)\left(\frac{1}{2}a+\frac{1}{2}b-\frac{3}{2}\cdot\frac{2}{3}m_c\right)\left(\frac{1}{2}a-\frac{1}{2}b+\frac{3}{2}\cdot\frac{2}{3}m_c\right)\left(-\frac{1}{2}a+\frac{1}{2}b+\frac{3}{2}\cdot\frac{2}{3}m_c\right)}\end{cases}$$

$$\begin{cases}S_1=\frac{1}{16}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}\\ S_1=\frac{1}{16}\sqrt{(a+b+2m_c)(a+b-2m_c)(a-b+2m_c)(-a+b+2m_c)}\end{cases}$$

Taking into account $\frac{S_1}{S}=\frac{1}{3}$, we get:

$$\begin{cases}4S=\frac{3}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}\\ 4S=\frac{3}{4}\sqrt{(a+b+2m_c)(a+b-2m_c)(a-b+2m_c)(-a+b+2m_c)}\end{cases} \quad (38)$$

Next, we perform similar calculations, as we did in part I of the calculations (with medians)

…

$$\left(\left(\frac{1}{2}a\right)^2-\left(\frac{1}{2}b\right)^2\right)^2+\left(\frac{4}{3}S\right)^2=\left(\frac{3}{2}\cdot\frac{2}{3}m_c\cdot\frac{1}{2}c\right)$$

$$\left(\frac{a^2-b^2}{4}\right)^2+\left(\frac{4}{3}S\right)^2=\left(m_c\cdot\frac{1}{2}c\right)$$

$$\left(\frac{3}{4}(a^2-b^2)\right)^2+(4S)^2=\left(\frac{3}{2}cm_c\right)^2$$

We have obtained the identities (37).

III PART OF THE CALCULATIONS.

In Part III of the calculations, we will work with the first identities from (36) and (37):

$$\begin{cases}(m_a^2-m_b^2)^2+(4S)^2=\left(\frac{3}{2}cm_c\right)^2\\ \left(\frac{3}{4}(a^2-b^2)\right)^2+(4S)^2=\left(\frac{3}{2}cm_c\right)^2\end{cases} \quad (39)$$

It can be seen that the identity system (39) contains all 7 components (14), (14*).

Taking into account (4), we know that if all medians (m_a, m_b, m_c) are integers, then all sides (a, b, c) must be even numbers. (39) let's write it like this:

$$\begin{cases}\left(\frac{m_a^2-m_b^2}{3}\right)^2+\left(\frac{4S}{3}\right)^2=\left(\frac{c}{2}m_c\right)^2\\ \left(\left(\frac{a}{2}\right)^2-\left(\frac{b}{2}\right)^2\right)^2+\left(\frac{4S}{3}\right)^2=\left(\frac{c}{2}m_c\right)^2\end{cases} \quad (40)$$

Taking into account (4), it can be seen that at least one of the sides (a, b, c) of the triangle is a multiple of 3 (in this case, let it be $a \equiv 0(mod3)$). Otherwise, all medians (m_a, m_b, m_c) will be multiples of 3. It is known that Heronian triangle is always divisible by 6. Therefore, it cannot be that only two sides of an integer triangle are divisible by 3, and one side is not divisible by 3. This follows from Heron's formula for finding the area of a triangle on its sides.

In continuation (14), we noticed that $c \equiv 0(mod2^k)$ and $k_{max} = 1$.

According to conditions (14) and (9), it is obvious that only one of the medians is divisible by 2. Let one of the two odd medians be m_c.

We know that every primitive Pythagorean triple arises from a unique pair of mutually prime numbers (from Euclid's formula).

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Remark. Suppose that in the identities (40) "Pythagorean triple is not primitive", and the following conditions are met for the natural number $p \neq 3$:

$$\frac{m_a^2-m_b^2}{3} \equiv 0(modp),\ \left(\frac{a}{2}\right)^2-\left(\frac{b}{2}\right)^2 \equiv 0(modp),\ \frac{4S}{3} \equiv 0(modp),\ \frac{c}{2}m_c \equiv 0(modp) \quad (40^*)$$

If (40*) is true, then we can assume that $c \equiv 0(modp)$. This means that we will be able to proportionally reduce the triangle ΔABC by $p \neq 3$ times. Therefore, all sides and medians of the triangle ΔABC decrease by $p \neq 3$ times, and the area $\left(\frac{4S}{3}\right)$ decreases by p^2 times (algebra requires that $\frac{c}{2}m_c \equiv 0(modp^2)$). This means that $m_c \equiv 0(modp)$ will be executed in (40*). Also, taking into account (40), there will be $m_a^2 - m_b^2 \equiv 0(modp^2)$. Under the conditions $c \equiv 0(modp),\ m_c \equiv 0(modp)$ and taking into account (4), both conditions $m_a^2 - m_b^2 \equiv 0(modp^2)$ and $m_a^2 + m_b^2 \equiv 0(modp^2)$ are satisfied. It turns out $m_a \equiv 0(modp), m_b \equiv 0(modp)$. And these conditions contradict the condition (14).

***

Are the Pythagorean triples in (40) primitive?

Taking into account the above, it turns out that the Pythagorean triples in (40) can be primitive only if

$m_c \neq 0(mod3)$. **(41)**

Assume that (41) is true, and write:

$$\frac{c}{2}m_c = \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 \quad (42)$$

$$\frac{4S}{3} = 2 \cdot \frac{a}{2} \cdot \frac{b}{2} \quad (43)$$

Taking into account that $S = \frac{3}{4}S_{\Delta ABC} = \frac{3}{4} \cdot \frac{1}{2}ab \cdot sin\angle ACB$, we write

$$\frac{c}{2}m_c = \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 \Rightarrow 2cm_c = a^2 + b^2 \quad (44)$$

$$\frac{4S}{3} = 2 \cdot \frac{a}{2} \cdot \frac{b}{2} \Rightarrow S = \frac{3}{4} \cdot \frac{1}{2}ab \Rightarrow \frac{3}{4} \cdot \frac{1}{2}ab = \frac{3}{4} \cdot \frac{1}{2}ab \cdot sin\angle ACB \Rightarrow$$

$$\Rightarrow sin\angle ACB = 1 \quad (45)$$

It turns out that $\angle ACB = \frac{\pi}{2}$, and the triangle ΔABC is rectangular. For a right triangle ΔABC, we write:

$$\begin{cases} a^2 + b^2 = c^2 \\ m_a^2 = \left(\frac{a}{2}\right)^2 + b^2 \\ m_b^2 = a^2 + \left(\frac{b}{2}\right)^2 \\ a^2 + b^2 = 2cm_c \end{cases} \quad (46)$$

It is known that if ΔABC is a right triangle, then (46) is not true [8] (*page 546, Theorem 2.7*).

It turns out that (41) is not true, and we agree

$$m_c \equiv 0(mod 3). \qquad (47)$$

It turns out that (41) is incorrect, Pythagorean triples (40) are not primitive, and according to the conditions between (14), (14*), (14**) the non-primitiveness of Pythagorean triples in (40) is due to $M = 3$.

Lemma 3 is proved.

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Proof of the theorem (by the method of infinite descent).

Suppose there is a triangle with integer area, medians and sides (Fig1). And the triangle ΔABC is the one with the smallest area among them.

Using the triangle ΔABC, we will construct the triangle ΔOA_0C.

Taking into account **Lemma 1**, **Lemma 2** and **Lemma 3** all medians of triangle ΔABC are multiples of 3. Consequently, the sides of triangle ΔOA_0C are integers. Since all sides of the triangle ΔABC are even (formulas (4)), then the medians of the triangle ΔOA_0C are integers. It is known from Lemma 1 (3) that

$$\frac{A_{\Delta OPC}}{A_{\Delta ABC}} = \frac{1}{3} \Rightarrow A_{\Delta OPC} = \frac{1}{3} A_{\Delta ABC} < A_{\Delta ABC} \qquad (48)$$

In other words, there is another triangle ΔOA_0C with integer sides, medians, and area less than the original triangle ΔABC. And this contradicts our assumption that the area of a triangle with integer area, medians and sides $A_{\Delta ABC}$ is the smallest.

By repeating this process will eventually yield an integral perfect triangle of area less than 1, which is impossible.

The theorem is proved.

There are no triangles with three whole sides, three whole medians, and an entire area.

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