IS THERE ANY SIMILARITY BETWEEN THE FUNCTIONAL EQUATIONS OF THE EF FUNCTION AND THE RIEMANN ZETA FUNCTION?

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ANNOTATION.

The Riemann zeta function or Euler–Riemann zeta function, denoted by the Greek letter ξ (zeta), is a mathematical function of a complex variable defined as

$$\xi(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

It is known that the Riemann zeta function has an analytical extension to the entire complex plane.

This article examines the coefficients of the functional equation of the ef-function. Two mathematical expressions are compared: the functional equation of the Riemann zeta-function (1) and the functional equation (2), we denote as the functional equation of the ef-function

$$\xi(z) = \xi(1 - z) \quad \textbf{(1)}$$

$$F(z) = F(1 - (1 + \varepsilon)z) \quad \textbf{(2)}$$

These two mathematical expressions differ in the coefficients of the complex variables z, where instead of (-1) the coefficient $-(1 + \varepsilon) \to -1,\ \varepsilon > 0$ is written. I assume that both (1) and (2) have complex zeros (3)

$$z = a + i\beta \quad \textbf{(3)}$$

we denote as nontrivial zeros the ef-functions, the real parts of which are positive (4)

$$\mathrm{Re}(z) > 0 \quad \textbf{(4)}$$

For the functional equation of the Riemann zeta-function, (5) holds

$$0 < \mathrm{Re}(z) < 1 \quad \textbf{(5)}$$

At the same time, the ef-function retains its analyticity.

THE MAIN AND ONLY PURPOSE OF THE ARTICLE

The main purpose of this article is to find the values of the real parts (a), and some property of the imaginary parts ($i\beta$) of nontrivial zeros of the ef function (if they exist).

Solution

Let $z_{11} = \alpha_{11} + i\beta_{11}$ be the "first" nontrivial zero of the ef-function. In (2), the coefficient $-(1+\varepsilon)$ is replaced by $-(1+\varepsilon) = k$. Taking into account (2), we write

$F(z_{11}) = F(kz_{11} + 1)$

Another zero appears $z_{12} = kz_{11} + 1$. Let's write it down

$z_{12} = kz_{11} + 1 = k \cdot (\alpha_{11} + i\beta_{11}) + 1 = (k\alpha_{11} + 1) + i\beta_{11}k$

Let's write down the "rotation" of zeros:

$$F(z_{12}) = F(kz_{12} + 1) = F\big(k\big((k\alpha_{11} + 1) + i\beta_{11}k\big) + 1\big) =$$
$$= F\big((\alpha_{11}k^2 + k + 1) + i\beta_{11}k^2\big)$$

$z_{13} = (\alpha_{11}k^2 + k + 1) + i\beta_{11}k^2$

$$F(z_{13}) = F(kz_{13} + 1) = F\big(k\big((\alpha_{11}k^2 + k + 1) + i\beta_{11}k^2\big) + 1\big) =$$
$$= F\big((\alpha_{11}k^3 + k^2 + k + 1) + i\beta_{11}k^3\big)$$

… … …

$z_{1n} = kz_{1(n-1)} + 1 = (\alpha_{11}k^{n-1} + k^{n-2} + k^{n-3} + \cdots + k + 1) + i\beta_{11}k^{n-1}$

Let's simplify it

$z_{1n} = \alpha_{11}k^{n-1} + (k^{n-2} + k^{n-3} + \cdots + k + 1) + i\beta_{11}k^{n-1}$

$$z_{1n} = \alpha_{11}k^{n-1} + \frac{k^{n-1}-1}{k-1} + i\beta_{11}k^{n-1} \quad \textbf{(6)}$$

Let's write down the $(n+1)$ th zero

$$z_{1(n+1)} = \alpha_{11}k^{n} + \frac{k^{n}-1}{k-1} + i\beta_{11}k^{n} \quad \textbf{(7)}$$

$(n-1)$ and n are consecutive natural numbers. So they are numbers of different parities. Let n be an even number and $(n-1)$ be a non-even number. In (6) and (7), we substitute $k = -(1+\varepsilon)$, and write:

$$z_{1n} = \alpha_{11}(-1-\varepsilon)^{n-1} + \frac{(-1-\varepsilon)^{n-1}-1}{(-1-\varepsilon)-1} + i\beta_{11}(-1-\varepsilon)^{n-1}$$

$$z_{1n} = -\alpha_{11}(1+\varepsilon)^{n-1} + \frac{-(1+\varepsilon)^{n-1}-1}{-2-\varepsilon} - i\beta_{11}(1+\varepsilon)^{n-1} \quad \textbf{(8)}$$

$$z_{1(n+1)} = \alpha_{11}(-1-\varepsilon)^{n} + \frac{(-1-\varepsilon)^{n}-1}{(-1-\varepsilon)-1} + i\beta_{11}(-1-\varepsilon)^{n}$$

$$z_{1(n+1)} = \alpha_{11}(1+\varepsilon)^{n} + \frac{(1+\varepsilon)^{n}-1}{-2-\varepsilon} + i\beta_{11}(1+\varepsilon)^{n} \quad \textbf{(9)}$$

Let's divide the imaginary part (8) by the imaginary part (9)

$$\lim_{\varepsilon\to 0}\frac{\mathrm{Im}(z_{1n})}{\mathrm{Im}(z_{1(n+1)})}=\lim_{\varepsilon\to 0}\frac{-i\beta_{11}(1+\varepsilon)^{n-1}}{i\beta_{11}(1+\varepsilon)^{n}}=\lim_{\varepsilon\to 0}\frac{-1}{1+\varepsilon}=-1 \quad \textbf{(10)}$$

Corollary 1. It follows from the last result (10) that if the functional equation of the ef-function (2) has two zeros, then the imaginary parts of these zeros are opposite numbers. This is indicated by the value of the limit (-1) in (10).

In (8) and (9), ε and n are functionally independent variables.
The value of n depends only on our choice. Let's write it so that ε and n will be functionally dependent:

$n=2p,\quad n-1=2p-1,\quad 2p=\frac{C}{\varepsilon^2}$

here, $C\geq 1$ is the minimum value bounded from the top, allowing (ensuring) the integrity of $2p=\frac{C}{\varepsilon^2}$. We will take into account in the real parts (8) and (9).

$$-\alpha_{11}(1+\varepsilon)^{2p-1}+\frac{-(1+\varepsilon)^{2p-1}-1}{-2-\varepsilon}=-\alpha_{11}\frac{(1+\varepsilon)^{2p}}{(1+\varepsilon)}+\frac{-\frac{(1+\varepsilon)^{2p}}{(1+\varepsilon)}-1}{-2-\varepsilon}=$$

$$=\frac{(1+\varepsilon)^{2p}}{(1+\varepsilon)}\left(-\alpha_{11}+\frac{-1-\frac{(1+\varepsilon)}{(1+\varepsilon)^{2p}}}{-2-\varepsilon}\right)=\frac{(1+\varepsilon)^{2p}}{(1+\varepsilon)}\left(-\alpha_{11}-\frac{1+\frac{(1+\varepsilon)}{(1+\varepsilon)^{2p}}}{-2-\varepsilon}\right)$$

$$\mathbf{Re}z_{1n}=\mathbf{Re}z_{1(2p)}=\frac{(1+\varepsilon)^{\frac{C}{\varepsilon^2}}}{1+\varepsilon}\left(-\alpha_{11}-\frac{1+\frac{1+\varepsilon}{(1+\varepsilon)^{\frac{C}{\varepsilon^2}}}}{-2-\varepsilon}\right) \quad \textbf{(11)}$$

$$\mathbf{Re}z_{1(n+1)}=\mathbf{Re}z_{1(2p+1)}=(1+\varepsilon)^{\frac{C}{\varepsilon^2}}\left(\alpha_{11}+\frac{1-\frac{1}{(1+\varepsilon)^{\frac{C}{\varepsilon^2}}}}{-2-\varepsilon}\right) \quad \textbf{(12)}$$

Now we divide the real part (8) by the real part (9). In other words, we divide (11) by (12)

$$\frac{\mathbf{Re}z_{1(2p)}}{\mathbf{Re}(z_{1(2p+1)})}=\frac{\frac{(1+\varepsilon)^{\frac{C}{\varepsilon^2}}}{1+\varepsilon}\left(-\alpha_{11}-\frac{1+\frac{1+\varepsilon}{(1+\varepsilon)^{\frac{C}{\varepsilon^2}}}}{-2-\varepsilon}\right)}{(1+\varepsilon)^{\frac{C}{\varepsilon^2}}\left(\alpha_{11}+\frac{1-\frac{1}{(1+\varepsilon)^{\frac{C}{\varepsilon^2}}}}{-2-\varepsilon}\right)}=\frac{\frac{1}{1+\varepsilon}\left(-\alpha_{11}-\frac{1+\frac{1+\varepsilon}{(1+\varepsilon)^{\frac{C}{\varepsilon^2}}}}{-2-\varepsilon}\right)}{\left(\alpha_{11}+\frac{1-\frac{1}{(1+\varepsilon)^{\frac{C}{\varepsilon^2}}}}{-2-\varepsilon}\right)} \quad \textbf{(13)}$$

We go to the limit and take into account $\lim_{\varepsilon\to 0}(1+\varepsilon)^{\frac{C}{\varepsilon^2}} = \lim_{\varepsilon\to 0}\left((1+\varepsilon)^{\frac{1}{\varepsilon}}\right)^{\frac{C}{\varepsilon}} \to \infty$

$$\lim_{\varepsilon\to 0}\frac{\mathbf{Re}z_{1(2p)}}{\mathbf{Re}\left(z_{1(2p+1)}\right)} = \lim_{\varepsilon\to 0}\frac{\frac{1}{1+0}\left(-\alpha_{11}-\frac{1+\frac{1+0}{\infty}}{-2-0}\right)}{\left(\alpha_{11}+\frac{1-\frac{1}{\infty}}{-2-0}\right)} = \frac{-\alpha_{11}+\frac{1}{2}}{\alpha_{11}-\frac{1}{2}} = \frac{-2\alpha_{11}+1}{2\alpha_{11}-1} \quad \mathbf{(14)}$$

There are two possible options for (14).

1. $2\alpha_{11}-1 \neq 0$. It turns out,

$$\lim_{\varepsilon\to 0}\frac{\mathbf{Re}z_{1(2p)}}{\mathbf{Re}\left(z_{1(2p+1)}\right)} = \frac{-2\alpha_{11}+1}{2\alpha_{11}-1} = -1$$

This contradicts conditions (4) and (5).

2. $2\alpha_{11}-1 = 0$. It turns out, $\alpha_{11} = \frac{1}{2}$.

COROLLARY 2. If the functional equation of an ef-function has complex zeros, then the real parts of such zeros are always $\mathbf{\frac{1}{2}}$.

P.S. Is (1) a special case of (2)?