PRIME NUMBER DISTRIBUTION THEOREM

THE 3RD LANDAU PROBLEM (LEGENDRE'S CONJECTURE)

BROCARD'S CONJECTURE

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SETTING THE TASK. The set of natural numbers $[1,(N+2)N]$ we will write in the form of a table, with N consecutive numbers in each row as follows (**In this article, we are not talking about the "sieve of Eratosthenes"**):

1	2	$3,\dots$	$N-1$	N
$N+1$	$N+2$	$N+3,\dots$	$2N-1$	$2N$
$2N+1$	$2N+2$	$2N+3,\dots$	$3N-1$	$3N$
$3N+1$	$3N+2$	$3N+3,\dots$	$4N-1$	$4N$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$mN+1$	$mN+2$	$mN+3,\dots$	$(m+1)N-1$	$(m+1)N$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$(N-1)N+1$	$(N-1)N+2$	$(N-1)N+3,\dots$	N^2-1	N^2
N^2+1	N^2+2	$N^2+3,\dots$	$(N+1)N-1$	$(N+1)N$
$(N+1)N+1$	$(N+1)N+2$	$(N+1)N+3,\dots$	$(N+2)N-1$	$(N+2)N$
$(N+2)N+1=(N+1)^2$	**In this article, we are not talking about the "sieve of Eratosthenes"**			

ANNOTATION. Using the prime number distribution theorem, we prove many open problems in number theory, such as Brocard's hypothesis, Landau's 3rd problem, and others.

Action. At the same time, in the randomly selected and first rows of the table, we cross out all the numbers that are multiples of the prime number $p \in L = \{2,3,5,\dots,P\}$. Here L is the set of all the primes in the first row of the table, and p runs through all the primes in the set L. In some rows of the table (before the deletion begins), the number of numbers that are multiples of some numbers (hereinafter in the text these numbers are designated as *critical numbers*) of the set L is 1 more than in the first row (hereinafter in the text these numbers are designated as *problem numbers*). In randomly selected rows, we cross out no more numbers than in the first row of the table. If necessary (**theoretically**), in order to maintain such a balance of the numbers crossed out (in the first rows taken at random), in some cases we leave the problematic numbers (**theoretically**) not crossed out. As a result, we prove (Lemma 3) that, according to the results of crossing out, there are actually no problem numbers left in the table. And in the first line the number "1" (one) remains uncrossed out. This means that at least one number, a prime number, remains uncrossed in each row of the table.

THEOREM. For any natural numbers $N \geq 2$ and k, where $1 \leq k \leq N+2$, there is at least one prime number in the range $[(k-1)N+1, kN]$.

IN OTHER WORDS: there is at least one prime number in each complete row of the above table.

PROOF OF THE THEOREM.

Obviously, there is always a prime number in the first row of the table.

ACCORDING TO BERTRAND'S POSTULATE: for any natural $N \geq 2$, there is a prime number in the interval $[N, 2N]$. Therefore, in this paper, we do not analyze the second line of the table for the presence of primes in it (we do not prove it).

Now let's prove that, **starting from the third row**, there is at least one prime number in an arbitrary row of the table.

LEMMA 1. For any (arbitrarily taken) natural number $m \leq N$, in an arbitrarily taken row of the table, the number of numbers $f(m)$ that are multiples of m is written:

$$f(m) = t(m) + \Delta_m \qquad \textbf{(1)}$$

Here $t(m) = \left[\frac{N}{m}\right]$ is the number of multiples of m in the first row of the table.

Let's prove that either $\Delta_m = 0$ or $\Delta_m = 1$.

PROOF OF LEMMA 1. It suffices to prove that $\Delta_m < 2$.

The length (the number of all numbers in a row) of the first row is written as follows:

$$N = (m-1) + \left(1 + \left(\left[\frac{N}{m}\right] - 1\right) \cdot m\right) + \alpha = \left[\frac{N}{m}\right] \cdot m + \alpha \qquad \textbf{(2)}$$

Here $0 \leq \alpha \leq m-1$ **(3)**

$(m-1)$ – the number of numbers at the beginning of the first line up to m,

α is the number of numbers (at the end of the first line) after the largest multiple of m.

Suppose that in some row of the table $\Delta_m = 2$. In this case, the **minimum** length (number of numbers) is there will be such a line:

$$N = \left(\left(\left[\frac{N}{m}\right] + \Delta_m\right) - 1\right) \cdot m + 1 = \left(\left(\left[\frac{N}{m}\right] + 2\right) - 1\right) \cdot m + 1 = \left[\frac{N}{m}\right] \cdot m + m + 1 \qquad \textbf{(4)}$$

Taking into account (2), (3), and (4), we obtain the following contradiction:

$$\left[\frac{N}{m}\right] \cdot m + \alpha = \left[\frac{N}{m}\right] \cdot m + m + 1 \Rightarrow \alpha = m + 1$$

LEMMA 1 *is proved.*

Designation. For $\Delta_m = 0$, we denote the number m as a *good number*. And for $\Delta_m = 1$, we denote the number m as a *critical number*.

Designation. If $\Delta_m = 1$ in any row, then we will denote this as follows: in this row, the number $f(m)$ is "*increased in favor of the number*" $\left[\frac{N}{m}\right] + 1$, or $f(m) \to \left[\frac{N}{m}\right] + 1$.

Designation. If $\Delta_m = 1$, then there is a number F in such a row that is a multiple of $m \cdot \left(\left[\frac{N}{m}\right] + 1\right) > N$. In other words, there was an increase in $f(m)$ in favor of $\left[\frac{N}{m}\right] + 1$ in this row, that is, $f(m) \to \left[\frac{N}{m}\right] + 1$. Let's denote the number F as a *problem number*, and define it this way

$$F = zm\left(\left[\frac{N}{m}\right] + 1\right) = zP_1P_2 = zP_1\left(\left[\frac{N}{P_1}\right] + 1\right) \geq N + 1. \qquad \textbf{(4A)}$$

Here $m = P_1$, $\left[\frac{N}{P_1}\right] + 1 = P_2$, $P_1 \to \left[\frac{N}{P_1}\right] + 1$, z is a natural number.

PROPERTY 1. Let's number the rows of the table as $1, 2, 3, \ldots$ For rows under numbers

$$\{1, m+1, 2m+1, 3m+1, \ldots\} \qquad \textbf{(4B)}$$

the value $\Delta_m = 0$ is periodically repeated.

Consequence of **PROPERTY 1**. In all the lines indicated by (4B), the number m is good.

LEMMA 2. Suppose that we crossed out in an arbitrary row (at the same time in the first row) of the table all the numbers that are multiples of the good (if any) prime number $p_1 \in L$, for which there was

$f(p_1) = t(p_1)$.

After such a deletion, we will study the number of remaining (non-deleted) numbers that are multiples of an arbitrarily taken prime number $p_i \in L \setminus p_1$, for which it was originally

$f(p_i) = \left[\frac{N}{p_i}\right] + \Delta_{p_i} = t(p_1) + \Delta_{p_1}$.

And after crossing out the numbers that are multiples of $p_1 \in L$, it becomes

$F(p_i) = T(p_i) + \delta_{p_i}$.

At the same time, it is obvious that in the first and randomly selected rows there will be no numbers that are multiples of $p_1 p_i$.

Let's prove that

$$\delta_{p_i} \leq \Delta_{p_i}. \quad \textbf{(5)}$$

PROOF OF LEMMA 2. According to (1), for a prime number p_i and for $m = p_1 p_i$, we write:

$$f(p_i) = t(p_i) + \Delta_{p_i} \quad \textbf{(6)}$$

$$f(p_1 p_i) \geq t(p_1 p_i) \quad \textbf{(7)}$$

$$F(p_i) = T(p_i) + \delta_{p_i} \quad \textbf{(8)}$$

From (6) we subtract (7)

$$f(p_i) - f(p_1 p_i) \leq t(p_i) - t(p_1 p_i) + \Delta_{p_i} \quad \textbf{(9)}$$

In (8) we replace

$F(p_i) = f(p_i) - f(p_1 p_i)$

$T(p_i) = t(p_i) - t(p_1 p_i)$

We get

$$F(p_i) \leq T(p_i) + \Delta_{p_i} \quad \textbf{(10)}$$

Compare (8) and (10), we get

$$\delta_{p_i} \leq \Delta_{p_i} \quad \textbf{(11)}$$

LEMMA 2 *is proved.*

Corollary 1 of **Lemma 2**. Good numbers do not become critical in the process of crossing out.

Corollary 2 of **Lemma 2**. If in an arbitrary string, for $\Delta_m = 1$, the number $\left[\frac{N}{m}\right] + 1$ (or one of its factors) is a good number, then when crossing out numbers that are multiples of the number $\left[\frac{N}{m}\right] + 1$ (or its good divisor), the number m becomes good. For example, for $N = 13$ in the third row $\Delta_3 = 1$. In other words, in the first row of such a table, four numbers $3, 6, 9, 12$ are multiples of 3, and in the third row there are five such numbers $27, 30, 33, 36, 39$. That is $f(3) \rightarrow \left[\frac{13}{3}\right] + 1 = 5$. The number 5 in this row is a good number, that is, $\Delta_5 = 0$. In the third line, we cross out two numbers $(30, 35)$ that are multiples of the good number 5. In parallel, and in the first line, we cross out two numbers $(5, 10)$ that are multiples of the good number 5. In the new state of the third row of the table, the number of numbers $(27, 33, 36, 39)$ that are multiples of 3 has become the same as in the first row $(3, 6, 9, 12)$ – four. That is, in the beginning there was $f(3) = \left[\frac{13}{3}\right] + 1 = t(3) + 1 = 4 + 1 = 5$. And after crossing out the numbers that are multiples of 5, for the number 3 it turned out

$\delta_3 = 0 \Rightarrow F(3) = T(3) + \delta_3 = T(3) + 0 = 4$

1	2	3	4	~~5~~	6	7	8	9	~~10~~	11	12	13
14	15	16	17	18	19	20	21	22	23	24	25	26
27	28	29	~~30~~	31	32	33	34	~~35~~	36	37	38	39

Corollary 3 of **Lemma 2**. At any stage of deletion **If** $\Delta_p = 0$ (or $\delta_p = 0$), then in an arbitrarily chosen row of the table we will delete no more numbers (multiples of good p) than in the first row of the table (multiples of p), and in this case in an arbitrarily chosen row there will not remain a single number multiple of p.

Lemma 3. **If** $\Delta_p = 1$, then there is a critical prime p in an arbitrary string, and there may be a problem number (4A)

$$F = zm\left(\left[\frac{N}{m}\right] + 1\right) = zP_1P_2 = zP_1\left(\left[\frac{N}{P_1}\right] + 1\right) \geq N + 1.$$

Let's prove that after crossing out (in an arbitrarily taken row, we cross out no more numbers than in the first row) numbers that are multiples of all the primes of the set $L = \{2,3,5, \dots, P\}$, there will not be a single problem number left in the table.

PROOF OF LEMMA 3. Proof by contradiction. In the proofs of individual lemmas, the letter γ indicates the values corresponding to the conditions of the lemma. Based on the results of the two lemmas, it is obvious that when crossing out numbers that are multiples of good numbers, no more numbers were crossed out in an arbitrary row than in the first row of the table. In the process of this crossing out, some critical numbers have become kind. Therefore, if at the end of the crossing out any problematic numbers remain uncrossed, then the multipliers (divisors) of such problematic numbers can only be critical primes (since numbers with good divisors have already been crossed out). Let's assume that at the end of the strikeout in an arbitrary line, some problematic numbers F remained uncrossed (4).

Let's make a table of possible such problematic numbers. Here $\{P_1, P_2, P_3, P_4\}$ are the set of all possible critical numbers:

$F_1 = P_1P_2P_3P_4$	$F_2 = P_1^3$	$F_3 = P_1P_2^2$	$F_4 = P_1P_2P_3$

AUXILIARY LEMMA 3.1. If the problem number that has not been crossed out has the form $F_1 = P_1P_2P_3P_4$, then according to (4), the inequalities are fulfilled for $\{P_\mu, P_j, P_y, P_r\} = \{P_1, P_2, P_3, P_4\}$:

$P_\mu \cdot P_j \geq N + 1, \quad P_y \cdot P_r \geq N + 1.$

Therefore,

$$F_1 = P_1P_2P_3P_4 \geq (N + 1)^2 \qquad \textbf{(12)}$$

(12) contradicts the assumption, since the number $(N + 1)^2$ is outside the table.

AUXILIARY LEMMA 3.1 *is proved*.

AUXILIARY LEMMA 3.2. If the problem number that has not been crossed out has the form $F_2 = P_1^3$, then one option is possible:

$P_1 \to P_1,$

Therefore,

$P_1 \to P_1 \Rightarrow P_1 = \left[\frac{N}{P_1}\right] + 1 \Rightarrow P_1 \cdot \left(\left[\frac{N}{P_1}\right] + 1\right) = P_1^2 \Rightarrow N + 1 \le P_1^2 < 2N$

Since in the second row of the table the number P_1^2 is the smallest multiple of P_1, we write $P_1^2 - P_1 < N$, and continue as follows:

$P_1^2 - N = \gamma < P_1 \Rightarrow \gamma \le P_1 - 1 \Rightarrow P_1\gamma \le P_1^2 - P_1 < N \Rightarrow P_1\gamma < N$,

and therefore

$$P_1^2 - N = \gamma \Rightarrow P_1^3 - P_1 N = P_1\gamma \Rightarrow P_1^3 = P_1 N + P_1\gamma \qquad \textbf{(13)}$$

(13) means that the number $F_2 = P_1^3 = P_1 N + P_1\gamma$ is in the $(P_1 + 1)$ th row of the table. According to the *corollary of* **PROPERTY 1**, the number P_1 is kind, which means that the number $F_2 = P_1^3$ is actually not problematic.

AUXILIARY LEMMA 3.2 *is proved.*

AUXILIARY LEMMA 3.3. If the problem number that has not been crossed out has the form $F_3 = P_1 P_2^2$, then there are four possible options:

$P_1 \to P_2 \to P_1$	$P_1 \to P_2 \to P_1$	$P_1 \to P_2 \to P_2$	$P_1 \to P_2 \to P_2$
$P_1 > P_2$	$P_1 < P_2$	$P_1 > P_2$	$P_1 < P_2$

If the condition $P_1 \to P_2 \to P_1$ and $P_1 > P_2$ is true **for** $F_3 = P_1 P_2^2$, then $P_1 P_2$ is the smallest number in the second row of the table that is a multiple of both P_1 and P_2. Next, for $\gamma < P_2 < P_1$, we write $P_1 P_2 = N + \gamma$. Multiply the latter by P_2, and we get

$F_3 = P_1 P_2^2 = P_2 N + P_2\gamma$

Since $\gamma < P_1$, then $P_2\gamma < N$. Otherwise, the number $P_2\gamma < P_1 P_2$ must be in the second row ($P_1 P_2$ is the smallest number in the second row that is a multiple of P_2). It turns out that the number $F_3 = P_1 P_2^2 = P_2 N + P_2\gamma$ is in the $(P_2 + 1)$ th row, therefore, according to the *corollary of* **PROPERTY 1**, the number P_2 is not critical, and the number $F_3 = P_1 P_2^2$ is not problematic.

SIMILARLY, for $P_1 \to P_2 \to P_1$ and $P_1 < P_2$, we prove that the number P_2 is not critical, and the number $F_3 = P_1 P_2^2$ is not problematic.

If for $F_3 = P_1 P_2^2$ we have $P_1 \to P_2 \to P_2$ and $P_1 > P_2$, then the number P_2^2 is the smallest number in the second row of the table that is a multiple of P_2. Let's write

$P_2^2 = N + \gamma$. Multiply by P_1 and we get $P_1P_2^2 = P_1N + P_1\gamma$. Taking into account $\gamma < P_2 < P_1$, we get $P_1\gamma < N$. The number $F_3 = P_1P_2^2 = P_1N + P_1\gamma$ is in the $(P_1 + 1)$ th row, therefore, according to the *corollary of* **PROPERTY 1**, the number P_1 is not critical, and the number $F_3 = P_1P_2^2$ is not problematic.

If the condition $P_1 \to P_2 \to P_2$ and $P_1 < P_2$ is true **for** $F_3 = P_1P_2^2$, then it turns out that the numbers P_1P_2 and P_2^2 are simultaneously the smallest numbers in the second row that are multiples of P_2. And this is not possible because of $P_1 \neq P_2$.

AUXILIARY LEMMA 3.3 *is proved*.

AUXILIARY LEMMA 3.4. If the uncrossed out problem number has the form $F_4 = P_1P_2P_3$, then there are three options

$P_1 \to P_2 \to P_3 \to P_1$	$P_1 \to P_2 \to P_3 \to P_2$	$P_1 \to P_2 \to P_3 \to P_3$

If **for** $F_4 = P_1P_2P_3$ the condition $P_1 \to P_2 \to P_3 \to P_1$ is met, then theoretically we get:

* P_1P_2 in the second row is the smallest number that is a multiple of P_1.

** P_3P_1 in the second row is the smallest number that is a multiple of P_3. That is, it should be $P_1P_2 < P_3P_1$.

*** P_2P_3 in the second row is the smallest number that is a multiple of P_2. That is, theoretically it should be:

$P_2P_3 < P_1P_2 \Rightarrow P_3 < P_1$,

$P_2P_3 > P_3P_1 \Rightarrow P_2 > P_1$

$P_1P_2 < P_3P_1 \Rightarrow P_2 < P_3$.

According to the first two inequalities, $P_3 < P_1 < P_2 \Rightarrow P_3 < P_2$ is obtained. And this contradicts the third inequality, where $P_2 < P_3$.

If **for** $F_4 = P_1P_2P_3$ the condition $P_1 \to P_2 \to P_3 \to P_2$ is fulfilled, then theoretically it turns out:

* P_1P_2 in the second row is the smallest number that is a multiple of P_1.

** P_2P_3 in the second row is the smallest number that is a multiple of P_2. Also, the condition $P_3 \to P_2$ means that the number P_2P_3 in the second row is the smallest multiple of P_3. That is, it should be $P_2P_3 < P_1P_2$.

Let's write $P_2P_3 = N + \gamma$. Multiply by P_1, we get $P_1P_2P_3 = P_1N + P_1\gamma$. Так как $\gamma < P_2,\ \gamma < P_3$, то $P_1\gamma < P_1P_2$. Since $\gamma < P_2,\ \gamma < P_3$, then $P_1\gamma < P_1P_2$. The latter means that the number $P_1\gamma$ is in the first row (P_1P_2 in the second row is the smallest multiple of P_1). It turns out that the number $F_4 = P_1P_2P_3 = P_1N + P_1\gamma$ is in the $(P_1 + 1)$ -th row, therefore, according to the *corollary of* **PROPERTY 1**, the number P_1 is not critical, and the number $F_4 = P_1P_2P_3$ is not problematic.

If the condition $P_1 \to P_2 \to P_3 \to P_3$ is true **for** $F_4 = P_1P_2P_3$, then theoretically it turns out:

* P_1P_2 in the second row is the smallest number that is a multiple of P_1.

** P_2P_3 in the second row is the smallest number that is a multiple of P_2. We get $P_2P_3 < P_1P_2$.

Let's write $P_2P_3 = N + \gamma \Rightarrow \gamma < P_2$. Multiply by P_1, we get $P_1P_2P_3 = P_1N + P_1\gamma$. Since $\gamma < P_2$, the number $P_1\gamma$ is in the first row (P_1P_2 in the second row is the smallest multiple of P_1).This means that the number $F_4 = P_1P_2P_3 = P_1N + P_1\gamma$ is in the $(P_1 + 1)$ th row, therefore, according to the corollary of PROPERTY 1, the number P_1 is not critical, and the number $F_4 = P_1P_2P_3$ is not problematic.

AUXILIARY LEMMA 3.4 *is proved.*

LEMMA 3 *is proved.*

THEOREM *is proved.*

COROLLARY 1. SOLUTION OF THE 3RD LANDAU PROBLEM (Legendre's conjecture). For any natural N between N^2 and $(N+1)^2$ there is at least one prime number**.**

It is obvious that Legendre's hypothesis is a special case of the prime number distribution theorem, and for any natural N between N^2 and $(N+1)^2$ there will be at least two primes, since there are two complete rows in the specified interval (at least one prime number in each).

COROLLARY 2. BROCARD'S CONJECTURE. For any natural number n between p_n^2 and p_{n+1}^2 (where $p_n > 2$ and p_{n+1} are two consecutive primes), there are at least four primes.

For any prime number $p_n > 2$, we can write as follows:

$p_n = N - 1$ and $p_n + 2 = N + 1$.

$p_{n+1} - p_n \geq 2$

Between $p_n^2 = (N-1)^2$ and $(p_n + 2)^2 = (N+1)^2$ there are four complete lines, each of which has at least one prime number. We take into account that the minimum difference between consecutive (starting from 3) primes is 2, and therefore we chose $p_{n+1} = N + 1$. So, the greater the difference between consecutive primes, the more primes there are between their squares.

		$\cdots$	$(N-2)N$
1	$(N-1)^2$	$\cdots$	$(N-1)N$
2	$(N-1)N+1$	$\cdots$	N^2
3	N^2+1	$\cdots$	$(N+1)N$
4	$(N+1)N+1$	$\cdots$	$(N+2)N$
	$(N+1)^2$	$\cdots$	