THERE IS NO HERON TRIANGLE WITH THREE RATIONAL MEDIANS

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ABSTRACT. The subject of this article is the proof that the Heron of a triangle with three integer medians does not exist. The article provides proofs of three lemmas. As a result, the method of infinite descent proved that the Heron of a triangle with three integer medians does not exist.
The relevance of this article lies in the fact that the problem under study is one of the unsolved problems of number theory.

KEY WORDS. Heron triangles; Integer triangle; Number theory; geometry.

CLASSIFICATION NUMBERS: MSC: 11R04, 14G99, 11D99

1. INTRODUCTION

The problem: *Does a triangle with integer sides, integer medians and integer area exist?* [1, 2, 3].
It is known that there are triangles with integer sides and medians. For example, the smallest of these triangles has sides and medians (136, 170, 174) and (158, 131, 127), respectively.
In this article, we prove the theorem that there is no triangle with three integer sides, three integer medians and an integer area. To do this, the following three lemmas with proofs are given at the beginning:

Lemma 1 *For any triangle with rational sides and medians, there is another, but not similar triangle with rational sides and medians.*

Lemma 2. *If at least one median of a triangle with integer sides and medians is a multiple of 3, then all its medians are multiples of 3.*

Lemma 3. *If we assume that there is a triangle with integer sides, medians and area, then at least one of its medians must be multiple of 3.*

Theorem. There is no Heron triangle with three integer medians.
Using the results of the above three lemmas, we prove the theorem by the method of infinite descent.

2. PROOFS

Proof of Lemma 1.

Here it is proved that triangles with integer sides and medians exist only and only in pairs (like "twins") – one (any) of which follows from the other, and these two triangles are not similar triangles between themselves.

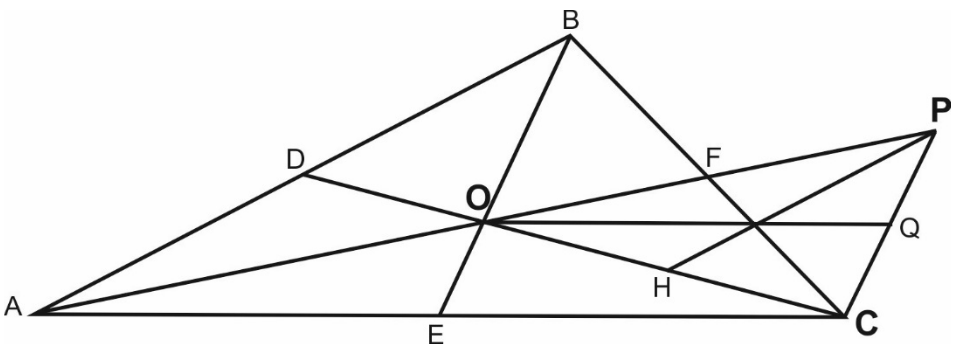

Fig1

Here (for Fig1): $AB = c,\ BC = a,\ AC = b,\ AF = m_a,\ BE = m_b,\ DC = m_c,$

$OP = \frac{2}{3}m_a,\ CP = \frac{2}{3}m_b, OC = \frac{2}{3}m_c\ \ OQ = \frac{1}{2}b,\ HP = \frac{1}{2}c, FC = \frac{1}{2}a$

The last six equalities are obtained by the results of Lemma 1.

Assume that the sides and medians of the triangle ΔABC are rational (Fig1). Using the triangle ΔABC, we construct the triangle ΔOPC.

To do this, draw (starting from point C) $CP \| OB$ to the intersection with the continuation of the median $AF = m_a$ at point P.

It turns out that the triangles ΔAPC and ΔAOE are similar and

$CP: EO = AP: AO = AC: AE = 2: 1$

Taking into account the properties of the medians ΔABC for the sides of the triangle ΔOPC, we obtain

$$CP = OB = \frac{2}{3}m_b, \quad OC = \frac{2}{3}m_c, \quad OP = OA = \frac{2}{3}m_a \tag{1}$$

For the medians of the triangle ABC, it turns out

$$FC = \frac{1}{2}BC = \frac{1}{2}a, \quad OQ = \frac{1}{2}AC = \frac{1}{2}b, \quad HP = \frac{1}{2}AB = \frac{1}{2}c \tag{2}$$

In other words, a triangle ΔOPC is constructed by parallel displacements of $\frac{2}{3}$ of the segments of the medians of triangle ΔABC. As for medians of triangle ΔOPC they are constructed by parallel displacements of $\frac{1}{2}$ parts of ΔABC triangle's sides. This means that all sides and medians of the ΔOPC triangle are also rational.

The triangles ΔABC and ΔOPC are not similar to each other. The sides of these triangles are rational and do not have the similarity property. The ratio of the areas of similar triangles should be equal to the square of the similarity coefficient.
In our case (Fig1) the ratio of the areas of the triangles is

$$\frac{A_{\Delta OPC}}{A_{\Delta ABC}} = \frac{1}{3} \tag{3}$$

which is not the square of rational number.

Lemma 1 is proved.

Note 1. Taking into account (3), we obtain equality (12).

Proof of Lemma 2.

Let's write down the formulas of dependence between the sides and medians of triangles:

$$\begin{cases} a = \frac{2}{3}\sqrt{2m_b^2 + 2m_c^2 - m_a^2} \\ b = \frac{2}{3}\sqrt{2m_a^2 + 2m_c^2 - m_b^2} \\ c = \frac{2}{3}\sqrt{2m_a^2 + 2m_b^2 - m_c^2} \end{cases} \Rightarrow \begin{cases} m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \\ m_b = \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2} \\ m_c = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2} \end{cases} \Rightarrow \begin{cases} m_a = \frac{1}{2}\sqrt{3b^2 + 3c^2 - (a^2 + b^2 + c^2)} \\ m_b = \frac{1}{2}\sqrt{3a^2 + 3c^2 - (a^2 + b^2 + c^2)} \\ m_c = \frac{1}{2}\sqrt{3a^2 + 3b^2 - (a^2 + b^2 + c^2)} \end{cases} \tag{4}$$

It is obvious from these three formulas that if at least one of the medians is a multiple of 3, then

$$a^2 + b^2 + c^2 \equiv 0 (mod 3) \tag{5}$$

This means that all three medians are multiples of 3.

Lemma 2 is proved.

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In addition to Fig1, we examine three more figures.

In the proof of Lemma 1, we have constructed the ΔOPC triangle (Fig1).

Using the ΔOPC triangle, three more triangles are constructed (the vertices of the ΔOPC triangle in all three figures are preserved and indicated in large letters).

Parameters of Fig2. Taking the vertex P as the intersection point of the medians, the triangle ΔOMC is constructed (Fig2).

It will be useful if we note that the ΔOMC triangle has one median (HM) equal to $\frac{3}{2}c$, two medians (OT, CV) equal to two medians of the ΔABC triangle (m_a, m_b). The two sides OM and CM of the triangle ΔOMC are not investigated in this article. If we construct a triangle from the medians $\left(\frac{3}{2}c, m_a, m_b\right)$ of the ΔOMC triangle, then the formula for the area of the resulting triangle (let's denote T_2) will be as follows

$$A_{T\_2} = \frac{1}{4}\sqrt{\left(m_a + m_b + \frac{3}{2}c\right)\left(m_a + m_b - \frac{3}{2}c\right)\left(m_a - m_b + \frac{3}{2}c\right)\left(-m_a + m_b + \frac{3}{2}c\right)} \quad (6)$$

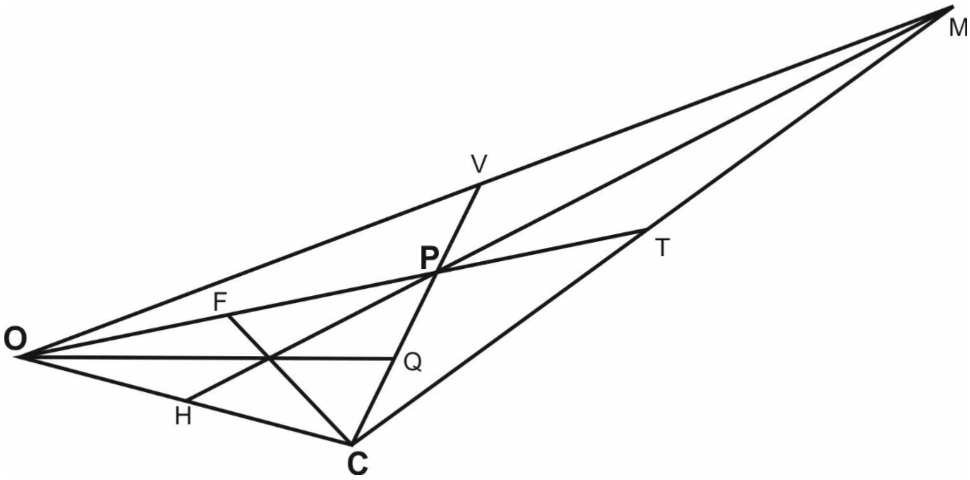

Fig2

Here (for Fig2): $CP = \frac{2}{3}m_b,\ CV = m_b,\ OP = \frac{2}{3}m_a,\ OT = m_a,\ OC = \frac{2}{3}m_c,$ $HM = \frac{3}{2}c,\ A_{\Delta OMC} = 3A_{\Delta OPC}$.

Parameters of Fig3. Taking the vertex O as the intersection point of the medians, the triangle ΔNPC is constructed (Fig3).

It will be useful if we note that the ΔNPC triangle has one median (NQ) equal to $\frac{3}{2}b$, two medians (PW, SU) equal to two medians of the ΔABC triangle (m_a, m_c). The two sides NP and NC of the triangle ΔNPC are not investigated in this article.

If we construct a triangle from the medians $\left(\frac{3}{2}b, m_a, m_c\right)$ of the ΔNPC triangle, then the formula for the area of the resulting triangle (let's denote T_3) will be as follows

$$A_{T\_3} = \frac{1}{4}\sqrt{\left(m_a + m_c + \frac{3}{2}b\right)\left(m_a + m_c - \frac{3}{2}b\right)\left(m_a - m_c + \frac{3}{2}b\right)\left(-m_a + m_c + \frac{3}{2}b\right)} \quad (7)$$

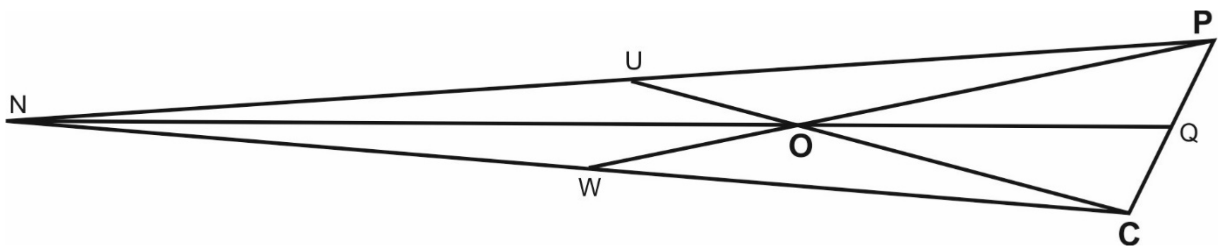

Fig3

Here (for Fig3): $CO = \frac{2}{3}m_c,\ CU = m_c,\ PO = \frac{2}{3}m_a,\ PW = m_a,\ PC = \frac{2}{3}m_b,$ $NQ = \frac{3}{2}b,\ A_{\Delta NPC} = 3A_{\Delta OPC}$.

Parameters of Fig4. Taking the vertex C as the intersection point of the medians, the triangle ΔOPL is constructed (Fig4).

It will be useful if we note that the ΔOPL triangle has one median (FL) equal to $\frac{3}{2}a$, two medians (PR, OK) equal to two medians of the ΔABC triangle (m_b, m_c).

The two sides OL and PL of the triangle ΔOPL are not investigated in this article.

If we construct a triangle from the medians $\left(\frac{3}{2}a, m_b, m_c\right)$ of the ΔNPC triangle, then the formula for the area of the resulting triangle (let's denote T_4) will be as follows

$$A_{T\_4} = \frac{1}{4}\sqrt{\left(m_b + m_c + \frac{3}{2}a\right)\left(m_b + m_c - \frac{3}{2}a\right)\left(m_b - m_c + \frac{3}{2}a\right)\left(-m_b + m_c + \frac{3}{2}a\right)} \quad (8)$$

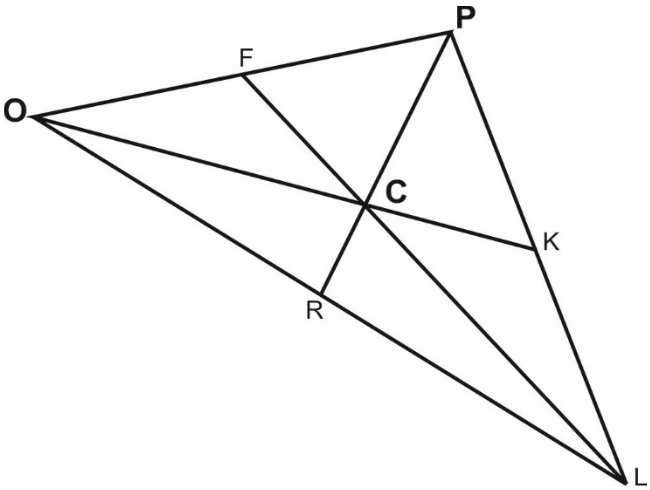

Fig4

Here (for Fig4): $PC = \frac{2}{3}m_b,\ PR = m_b,\ OC = \frac{2}{3}m_c,\ OK = m_c,\ OP = \frac{2}{3}m_a,$

$FL = \frac{3}{2}a,\ A_{\Delta OPL} = 3A_{\Delta OPC}$.

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If we construct a triangle from the medians (m_a, m_b, m_c) of the ΔABC triangle, then the formula for the area of the resulting triangle (let's denote T_1) will be as follows

$$A_{T\_1} = \frac{1}{4}\sqrt{(m_a + m_b + m_c)(m_a + m_b - m_c)(m_a - m_b + m_c)(-m_a + m_b + m_c)} \quad (9)$$

Let's write down the formula for the area of the triangle ΔABC.

$$A_{\Delta ABC} = \frac{1}{3}\sqrt{(m_a + m_b + m_c)(m_a + m_b - m_c)(m_a - m_b + m_c)(-m_a + m_b + m_c)} \quad (10)$$

Taking into account (9) and (10), we get

$$A_{T\_1} = \frac{3}{4}A_{\Delta ABC} \quad (11)$$

After studying Lemma 1, we learn that

$$A_{\Delta OMC} = A_{\Delta NPC} = A_{\Delta OPL} = A_{\Delta ABC} = 3A_{\Delta OPC} \quad (12)$$

Taking into account (11) and (12), we get

$$A_{T\_2} = A_{T\_3} = A_{T\_4} = A_{T\_1} \tag{13}$$

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Proof of Lemma 3.

Suppose there is a triangle ΔABC with integer sides, medians and area (Fig1), where

$$(a, b, c, m_a, m_b, m_c, A_{\Delta ABC}) = 1. \tag{14}$$

Let's prove that the medians of the triangle ΔABC are multiples of 3.

From two equations (9) and (6) we will make a system of equations.

$$\begin{cases} A_{T\_1} = \frac{1}{4}\sqrt{(m_a + m_b + m_c)(m_a + m_b - m_c)(m_a - m_b + m_c)(-m_a + m_b + m_c)} \\ A_{T\_2} = \frac{1}{4}\sqrt{\left(m_a + m_b + \frac{3}{2}c\right)\left(m_a + m_b - \frac{3}{2}c\right)\left(m_a - m_b + \frac{3}{2}c\right)\left(-m_a + m_b + \frac{3}{2}c\right)} \end{cases} \tag{15}$$

Taking into account (13), we denote

$$A_{T\_1} = A_{T_2} = S$$

$$\begin{cases} S = \frac{1}{4}\sqrt{(m_a + m_b + m_c)(m_a + m_b - m_c)(m_a - m_b + m_c)(-m_a + m_b + m_c)} \\ S = \frac{1}{4}\sqrt{\left(m_a + m_b + \frac{3}{2}c\right)\left(m_a + m_b - \frac{3}{2}c\right)\left(m_a - m_b + \frac{3}{2}c\right)\left(-m_a + m_b + \frac{3}{2}c\right)} \end{cases}$$

$$\begin{cases} 4S = \sqrt{(m_a + m_b + m_c)(m_a + m_b - m_c)(m_a - m_b + m_c)(-m_a + m_b + m_c)} \\ 4S = \sqrt{\left(m_a + m_b + \frac{3}{2}c\right)\left(m_a + m_b - \frac{3}{2}c\right)\left(m_a - m_b + \frac{3}{2}c\right)\left(-m_a + m_b + \frac{3}{2}c\right)} \end{cases}$$

$$\begin{cases} (4S)^2 = (m_a + m_b + m_c)(m_a + m_b - m_c)(m_a - m_b + m_c)(-m_a + m_b + m_c) \\ (4S)^2 = \left(m_a + m_b + \frac{3}{2}c\right)\left(m_a + m_b - \frac{3}{2}c\right)\left(m_a - m_b + \frac{3}{2}c\right)\left(-m_a + m_b + \frac{3}{2}c\right) \end{cases}$$

$$\begin{cases} (4S)^2 = ((m_a + m_b) + m_c)((m_a + m_b) - m_c)(m_c + m_a - m_b)(m_c - m_a + m_b) \\ (4S)^2 = \left((m_a + m_b) + \frac{3}{2}c\right)\left((m_a + m_b) - \frac{3}{2}c\right)\left(\frac{3}{2}c + m_a - m_b\right)\left(\frac{3}{2}c - m_a + m_b\right) \end{cases}$$

$$\begin{cases} (4S)^2 = ((m_a + m_b) + m_c)((m_a + m_b) - m_c)(m_c + (m_a - m_b))(m_c - (m_a - m_b)) \\ (4S)^2 = \left((m_a + m_b) + \frac{3}{2}c\right)\left((m_a + m_b) - \frac{3}{2}c\right)\left(\frac{3}{2}c + (m_a - m_b)\right)\left(\frac{3}{2}c - (m_a - m_b)\right) \end{cases}$$

$$\begin{cases} (4S)^2 = ((m_a + m_b)^2 - m_c^2)(m_c^2 - (m_a - m_b)^2) \\ (4S)^2 = \left((m_a + m_b)^2 - \left(\frac{3}{2}c\right)^2\right)\left(\left(\frac{3}{2}c\right)^2 - (m_a - m_b)^2\right) \end{cases} \tag{16}$$

Let's denote some expressions as follows (for convenience of calculations):

$$\begin{cases}(m_a+m_b)^2-m_c^2=x\\ m_c^2-(m_a-m_b)^2=y\\ \left(\frac{3}{2}c\right)^2-m_c^2=\Delta\\ (m_a+m_b)^2-\left(\frac{3}{2}c\right)^2=x-\Delta\\ \left(\frac{3}{2}c\right)^2-(m_a-m_b)^2=y+\Delta\end{cases} \tag{17}$$

Replace in (16)

$$\begin{cases}(4S)^2=xy\\ (4S)^2=(x-\Delta)(y+\Delta)\end{cases} \tag{18}$$

We get

$$xy=(x-\Delta)(y+\Delta)\Rightarrow xy=xy+x\Delta-\Delta y-\Delta^2\Rightarrow 0=(x-y-\Delta)\Delta \tag{19}$$

Let's solve equation (19)

Either $\Delta=0$, (20)

either $x-y-\Delta=0$. (21)

If $\Delta=0$, then $\Delta=\left(\frac{3}{2}c\right)^2-m_c^2=0$.

$$\left(\frac{3}{2}c\right)^2-m_c^2=0\Rightarrow\frac{3}{2}c-m_c=0\Rightarrow c=\frac{2}{3}m_c\Rightarrow\frac{2}{3}m_c=\frac{2}{3}\sqrt{2m_a^2+2m_b^2-m_c^2}\Rightarrow$$

$$\Rightarrow m_c=\sqrt{2m_a^2+2m_b^2-m_c^2}\Rightarrow m_c^2=2m_a^2+2m_b^2-m_c^2\Rightarrow$$

$$\Rightarrow m_c^2=m_a^2+m_b^2 \tag{22}$$

And this (22) is impossible. Because we took T_1 and T_2 in (15) arbitrarily. We could take T_1 and T_3, or T_1 and T_4.

If we took T_1 and T_3, we would get $m_b^2=m_a^2+m_c^2$.

If we took T_1 and T_4, we would get $m_a^2=m_b^2+m_c^2$.

If

$$\begin{cases}m_c^2=m_a^2+m_b^2\\ m_b^2=m_a^2+m_c^2,\\ m_a^2=m_b^2+m_c^2\end{cases}$$

then $m_a=m_b=m_c=0$.

If in (19) $x-y-\Delta=0$, then $y=x-\Delta$. (23)

Taking into account (23) in the first equation (18) we will replace $y=x-\Delta$, and in the second equation we will replace $x-\Delta=y$.

$$\begin{cases}(4S)^2 = xy \\ (4S)^2 = (x-\Delta)(y+\Delta)\end{cases} \Rightarrow \begin{cases}(4S)^2 = x(x-\Delta) \\ (4S)^2 = y(y+\Delta)\end{cases} \tag{24}$$

As a result (24) and (17) instead of the system of equations (16), we get the following system of equations (25).

$$\begin{cases}(4S)^2 = ((m_a+m_b)^2 - m_c^2)\left((m_a+m_b)^2 - \left(\frac{3}{2}c\right)^2\right) \\ (4S)^2 = (m_c^2 - (m_a-m_b)^2)\left(\left(\frac{3}{2}c\right)^2 - (m_a-m_b)^2\right)\end{cases} \tag{25}$$

$$\begin{cases}((m_a+m_b)^2 - m_c^2)\left((m_a+m_b)^2 - \left(\frac{3}{2}c\right)^2\right) - (4S)^2 = 0 \\ (m_c^2 - (m_a-m_b)^2)\left(\left(\frac{3}{2}c\right)^2 - (m_a-m_b)^2\right) - (4S)^2 = 0\end{cases}$$

$$\begin{cases}(m_a+m_b)^4 - (m_a+m_b)^2\left(\frac{3}{2}c\right)^2 - m_c^2(m_a+m_b)^2 + m_c^2\left(\frac{3}{2}c\right)^2 - (4S)^2 = 0 \\ m_c^2\left(\frac{3}{2}c\right)^2 - m_c^2(m_a-m_b)^2 - (m_a-m_b)^2\left(\frac{3}{2}c\right)^2 + (m_a-m_b)^4 - (4S)^2 = 0\end{cases}$$

$$\begin{cases}(m_a+m_b)^4 - \left(m_c^2 + \left(\frac{3}{2}c\right)^2\right)(m_a+m_b)^2 + \left(\frac{3}{2}cm_c\right)^2 - (4S)^2 = 0 \\ (m_a-m_b)^4 - \left(m_c^2 + \left(\frac{3}{2}c\right)^2\right)(m_a-m_b)^2 + \left(\frac{3}{2}cm_c\right)^2 - (4S)^2 = 0\end{cases} \tag{26}$$

Let's denote it like this

$$\begin{cases}(m_a+m_b)^2 = z \\ (m_a-m_b)^2 = w\end{cases} \tag{27}$$

We get two equivalent equations:

$$z^2 - \left(m_c^2 + \left(\frac{3}{2}c\right)^2\right)z + \left(\frac{3}{2}cm_c\right)^2 - (4S)^2 = 0 \tag{28}$$

$$w^2 - \left(m_c^2 + \left(\frac{3}{2}c\right)^2\right)w + \left(\frac{3}{2}cm_c\right)^2 - (4S)^2 = 0 \tag{29}$$

The first solution:

Is such equality possible?

$$z = w \Rightarrow (m_a+m_b)^2 = (m_a-m_b)^2 \Rightarrow$$

$$\Rightarrow (m_a+m_b)^2 - (m_a-m_b)^2 = 0 \Rightarrow 4m_am_b = 0 \tag{30}$$

According to the conditions of the problem $m_am_b \neq 0$.

The second solution (this method is useful for detailed analysis):

In this case, it is enough to solve one of the two equations (let's explore z). Obviously, we have a quadratic equation with coefficients

$$m_c^2 + \left(\frac{3}{2}c\right)^2 \text{ и } \left(\frac{3}{2}cm_c\right)^2 - (4S)^2, \tag{31}$$

and by requirement (26) has two roots $(m_a+m_b)^2$ and $(m_a-m_b)^2$.

That is, if

$$z^2-\left(m_c^2+\left(\frac{3}{2}c\right)^2\right)z+\left(\frac{3}{2}cm_c\right)^2-(4S)^2=0, \tag{32}$$

then $z_1=(m_a+m_b)^2$ и $z_2=(m_a-m_b)^2$. (33)

By Vieta 's theorem (the sum of the roots)

$$(m_a+m_b)^2+(m_a-m_b)^2=m_c^2+\left(\frac{3}{2}c\right)^2\Rightarrow$$

$$\Rightarrow 2m_a^2+2m_b^2-m_c^2=\frac{9}{4}c^2\Rightarrow$$

$$\Rightarrow c=\frac{2}{3}\sqrt{2m_a^2+2m_b^2-m_c^2} \qquad \underline{\textbf{(identity)}} \tag{34}$$

The last formula (4) – the formula (identity) of the sides of the triangle through the medians.

By Vieta 's theorem (the product of roots)

$$(m_a+m_b)^2(m_a-m_b)^2=\left(\frac{3}{2}cm_c\right)^2-(4S)^2\Rightarrow$$

$$\Rightarrow (m_a^2-m_b^2)^2+(4S)^2=\left(\frac{3}{2}cm_c\right)^2 \tag{35}$$

Obviously, if c, m_a, m_b, m_c, S are integers, then the numbers $m_a^2-m_b^2,\ 4S,\ \frac{3}{2}cm_c$ are a Pythagorean triple. If $\frac{3}{2}cm_c\equiv 0(mod3)$, then the Pythagorean triple $m_a^2-m_b^2,\ 4S,\ \frac{3}{2}cm_c$ not primitive. *There are three options*:

The first option. As a result, we get

$$m_a\equiv 0(mod3), m_b\equiv 0(mod3). \tag{36}$$

The second option. Formulas (36) are not correct. There are natural numbers v and w for which (37), (38) and (39) holds

$m_a^2-m_b^2=v^2-w^2$, where $v\equiv 0(mod3),\ w\equiv 0(mod3)$ (37)

$$2vw=4S \tag{38}$$

$$v^2+w^2=\frac{3}{2}cm_c \tag{39}$$

Taking into account (37) and (39) we get:

$$v^2+w^2\equiv 0(mod9)\Rightarrow\frac{3}{2}cm_c\equiv 0(mod9)\Rightarrow\frac{1}{2}cm_c\equiv 0(mod3) \tag{40}$$

Therefore, either $c\equiv 0(mod3)$ or $m_c\equiv 0(mod3)$. *(or both)* (41)

Let $c\equiv 0(mod3)$. (42)

Due to the arbitrariness of c and m_c, we get:

$$\begin{cases} c \equiv 0(mod3) \\ a \equiv 0(mod3) \\ b \equiv 0(mod3) \end{cases} \quad \text{taking into account (4)} \quad \begin{cases} m_c \equiv 0(mod3) \\ m_a \equiv 0(mod3) \\ m_b \equiv 0(mod3) \end{cases} \tag{43}$$

The third option. Formulas (36) are not correct. There are natural numbers v and w for which (44), (45) and (46) holds

$$m_a^2 - m_b^2 = 2vw, \text{ where } v \equiv 0(mod3),\ w \equiv 0(mod3) \tag{44}$$

$$v^2 + w^2 = \frac{3}{2}cm_c \tag{45}$$

$$v^2 - w^2 = 4S \tag{46}$$

Taking into account (44) and (45) we get:

$$v^2 + w^2 \equiv 0(mod9) \Rightarrow \frac{3}{2}cm_c \equiv 0(mod9) \Rightarrow \frac{1}{2}cm_c \equiv 0(mod3) \tag{47}$$

Therefore, either $c \equiv 0(mod3)$ or $m_c \equiv 0(mod3)$. *(or both)* (48)

Let $c \equiv 0(mod3)$. (49)

Due to the arbitrariness of c and m_c, we get:

$$\begin{cases} c \equiv 0(mod3) \\ a \equiv 0(mod3) \\ b \equiv 0(mod3) \end{cases} \quad \text{taking into account (4)} \quad \begin{cases} m_c \equiv 0(mod3) \\ m_a \equiv 0(mod3) \\ m_b \equiv 0(mod3) \end{cases} \tag{50}$$

Lemma 3 is proved.

Proof of the theorem (by the method of infinite descent).

Suppose there is a triangle with integer area, medians and sides (Fig1). And the triangle ΔABC is the one with the smallest area among them.

Using the triangle ΔABC, we will construct the triangle ΔOA_0C.

Taking into account **Lemma 1**, **Lemma 2** and **Lemma 3** all medians of triangle ΔABC are multiples of 3. Consequently, the sides of triangle ΔOA_0C are integers. Since all sides of the triangle ΔABC are even (formulas (4)), then the medians of the triangle ΔOA_0C are integers. It is known from Lemma 1 (3) that

$$\frac{A_{\Delta OPC}}{A_{\Delta ABC}} = \frac{1}{3} \Rightarrow A_{\Delta OPC} = \frac{1}{3}A_{\Delta ABC} < A_{\Delta ABC} \tag{51}$$

In other words, there is another triangle ΔOA_0C with integer sides, medians, and area less than the original triangle ΔABC. And this contradicts our assumption that the area of a triangle with integer area, medians and sides $A_{\Delta ABC}$ is the smallest.

By repeating this process will eventually yield an integral perfect triangle of area less than 1, which is impossible.

The theorem is proved.

There are no triangles with three whole sides, three whole medians, and an entire area.

LIST OF LITERATURE

1. Richard K. Guy, *Unsolved Problems in Number Theory, second edition.* Springer Verlag. New York: 1994. Page 188.
2. http://cage.ugent.be/~hvernaev/problems/Proble22.html
3. https://en.wikipedia.org/wiki/Heronian_triangle